

# PERMUTATION WEIGHTS FOR AFFINE LIE ALGEBRAS

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## Abstract

We show that permutation weights, which are previously introduced for finite Lie algebras, can be appropriately defined also for affine Lie algebras. This allows us to classify all the weights of an affine Weyl orbit explicitly. Let  $\Lambda$  be a dominant weight of an affine Lie algebra  $G_N^{(r)}$  for  $r=1,2,3$ . At each and every order  $M$  of weight depths, the set  $\wp_M(\Lambda)$  of permutation weights is formed out of a finite number of dominant weights of the finite Lie algebra  $G_N$ . In case of  $A_N^{(1)}$  algebras, we give the rules to determine the elements of a  $\wp_M(\Lambda)$  completely.

As being a positive test of our proposal, we consider the problem of calculating weight multiplicities for affine Lie algebras and hence our discussions are based on explicit computations of Weyl-Kac character formula. It is known that weight multiplicities are provided by string functions which are defined to be formal power series  $\sum_{M=0}^{\infty} C(M) q^M$  where the order  $M$  specifies the **depth** of weights contributing to  $C(M)$ . In the conventional calculational schemes which are based on Kac-Peterson form of affine Weyl groups, Weyl-Kac formula includes a sum over a part of the whole root lattice and hence it is seen that the roots of the same length contribute in general to  $C(M)$  for several values of  $M$ . On the contrary, we will determine, for any fixed value of  $M$ , the complete set of weights having depth  $M$  and contributing only to  $C(M)$ .

For applications of Weyl-Kac formula, one must also know the signatures which correspond to weights within the Weyl orbits of strictly dominant weights. This is given by the aid of a properly defined index.

Another emphasis is that the way of discussion adopted here gives us a possibility for extensions to other infinite dimensional Lie algebras beyond affine Lie algebras.

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## I. INTRODUCTION

It is well-known that the Weyl group of a Lie algebra plays an essential role in understanding of the structure of this algebra and also all of its representations [1]. In high energy physics, Weyl groups also enter in many areas such as calculation of fusion coefficients [2] or construction of exactly solvable models [3]. The structure of Weyl groups is completely known for finite Lie algebras in principle. In practice however, the problem is not so trivial even for finite Lie algebras, especially for the ones with some higher rank. The order of  $E_8$  Weyl group is, for instance, 696729600 and any application of Weyl character formula needs for  $E_8$  an explicit calculation of a sum over 696729600 Weyl reflections. We have shown in a recent work [4] that there is a way to overcome this difficulty.

For infinite dimensional Lie algebras, on the other hand, the situation is more complex because their Weyl groups are also infinite dimensional. Thanks to the seminal work of Kac and Peterson [5], it is known that affine Weyl groups can be expressed in the form of semi-direct products of a kind of translations and finite Weyl groups. It is commonly known, however, that there is a lack of knowledge concerning Weyl groups of infinite dimensional Lie algebras beyond affine Lie algebras.

An efficient way to describe the structure of a Weyl group is to study its action on the whole weight lattice. In many respects, this requires an explicit description of weights participating in the Weyl orbits. To this end, the Weyl character formula [6] establishes a principal example. We will concern ourselves here mainly with the calculation of characters by the aid of this formula. For a dominant weight  $\Lambda^+$ , it is known that the character  $Ch(\Lambda^+)$  of its irreducible representation  $R(\Lambda^+)$  is defined by

$$Ch(\Lambda^+) \equiv \sum_{\lambda^+} \sum_{\mu \in W(\lambda^+)} m_{\Lambda^+}(\mu) e^\mu \quad (I.1)$$

where the first sum is over  $\Lambda^+$  and all of its sub-dominant weights and  $m_{\Lambda^+}(\mu)$  's are multiplicities which count the number of times a weight  $\mu$  is repeated for  $R(\Lambda^+)$ . Note here that multiplicities are invariant under Weyl group actions and hence it is sufficient to determine only  $m_{\Lambda^+}(\lambda^+)$  for the whole Weyl orbit  $W(\lambda^+)$ . The crucial point here is however to calculate the multiplicities as positive integers. For this, the significant formula of Weyl says that

$$Ch(\Lambda^+) = \frac{A(\Lambda^{++})}{A(\rho)} \quad (I.2)$$

where  $\Lambda^{++}$  is the strictly dominant weight defined by

$$\Lambda^{++} \equiv \rho + \Lambda^+ \quad (I.3)$$

and  $\rho$  here is the Weyl vector which is also a strictly dominant weight. The object  $A(\Lambda^{++})$  covers by definition a sum over the whole Weyl group of underlying Lie algebra. As we have shown for finite Lie algebras [7], such a definition can always be replaced by the formula

$$A(\Lambda^{++}) \equiv \sum_{\mu \in W(\Lambda^{++})} \epsilon(\mu) e^\mu \quad (I.4)$$

where  $W(\Lambda^{++})$  is the Weyl orbit of  $\Lambda^{++}$ . Note here that, in the classical formulation, signatures are defined for Weyl reflections though, in (I.4),  $\epsilon(\mu)$  is the extended signature which can be defined properly for any weight  $\mu \in W(\Lambda^{++})$ . Formal exponentials  $e^\mu$  in (I.4) are defined just as in the book of Kac [8]. We will show in this paper that (I.4) is also valid for affine Lie algebras.

For finite Lie algebras, the equality of (I.1) and (I.2) is successful to bring us the multiplicities as positive integers. It is immediately seen however that, without a modification, this is not possible for affine Lie algebras. Let us define **string functions** by

$$C_{\Lambda^+}(\mu) \equiv \sum_{M=M_0(\mu, \Lambda^+)} e^{-\delta M} c_{\mu, \Lambda^+}(M) \quad (I.5)$$

where  $M_0(\mu, \Lambda^+)$  is the lowest depth for a sub-dominant weight  $\lambda^+$  for which  $\mu \in W(\lambda^+)$ . From discussions above, we know that  $M_0(\Lambda^+, \Lambda^+) \equiv 0$ . The basic observation for affine Lie algebras is to replace, in (I.1),  $C_{\Lambda^+}(\mu)$  instead of  $m_{\Lambda^+}(\mu)$ . This allows us to calculate the coefficients  $c_{\mu, \Lambda^+}(M)$  as positive integers. With this modification, it is known that Weyl formula turns to Weyl-Kac formula.

- It will be seen that answers can be conveniently formulated in terms of **fundamental weights** [10] (\*) which we have previously introduced as being a convenient basis for weight lattices of finite Lie algebras. We will see in this work that the use of fundamental weights prove useful also for infinite dimensional Lie algebras.

$A_N$  chain ( $N = 1, 2, \dots$ ) is in fact the essential key to understand the structure of all other finite Lie algebras in the Cartan classification. The same is also true for affine Lie algebras and hence we will proceed from the now on in the framework of  $A_N^{(1)}$  with the following Dynkin diagram:

We refer to excellent books of Kac [8] and Humphreys [11] for, respectively, affine and finite Lie algebras though some notation which we need in the sequel will also be given here. We define the fundamental weights  $\mu_I$  ( $I=1,2,.. N+1$ ) of  $A_N$  by

or, conversely, by

where  $\bar{\lambda}_i$ 's are fundamental dominant weights and  $\alpha_i$ 's are simple roots of  $A_N$ . By definition,  $A_N$  fundamental weights are constrained by

and they provide us a symmetric scalar product

The system of fundamental weights can be extended by adjoining the elements  $\Lambda_0$  ,  $\alpha_0$  and  $\delta$  with the adopted symmetric scalar products given below:

(\*) there must be no confusion between fundamental weights  $\mu_I$  and fundamental dominant weights  $\Lambda_\nu$ 's or  $\lambda_i$ 's

It is thus seen that, in view of (II.4) and (II.5), the definitions

$$\begin{aligned}
\alpha_0 &= \mu_{N+1} - \mu_1 + \delta \\
\alpha_1 &= \mu_1 - \mu_2 \\
\alpha_2 &= \mu_2 - \mu_3 \\
&\dots \\
\alpha_N &= \mu_N - \mu_{N+1}
\end{aligned} \tag{II.6}$$

give us a system of simple roots  $\{\alpha_\nu, \nu = 0, 1, \dots, N\}$  for  $A_N^{(1)}$  with the Dynkin diagram given above. Corresponding set  $\{\Lambda_\nu\}$  of affine fundamental dominant weights can then be given by

$$\Lambda_i = \Lambda_0 + \bar{\Lambda}_i, \quad i = 1, 2, \dots, N. \tag{II.7}$$

together with  $\Lambda_0$ . In the specification of an affine weight  $\Lambda$ , we have two central concepts of **level**  $k$  and **depth**  $M$  which are defined by

$$\begin{aligned}
k &\equiv (\lambda, \delta), \\
M &\equiv (\lambda, \Lambda_0).
\end{aligned} \tag{II.8}$$

It is known that level can always be defined to be a positive integer and, as a result of Schur lemma, it is constant for irreducible representations of affine Lie algebras. Beyond affine Lie algebras, any value of  $k$  will contribute in the same irreducible representation. The depth  $M$ , on the other hand, can be conveniently defined to be zero for the principal dominant weight  $\Lambda^+$  of an irreducible affine representation  $R(\Lambda^+)$  while it is in general a non-positive integer for its sub-dominant weights. (\*)

For any affine dominant weight  $\Lambda^+$ , its Weyl orbit  $W(\Lambda^+)$  is formed out of all weights  $\sigma(\Lambda^+)$  where  $\sigma$  is an affine Weyl reflection for which the decomposition

$$\sigma(\Lambda^+) = k \Lambda_0 + M \delta + \bar{\mu} \tag{II.9}$$

is always valid on condition that

$$\Lambda^+ \equiv k \Lambda_0 + \bar{\Lambda}^+ \tag{II.10}$$

and also

$$(\bar{\mu}, \bar{\mu}) - (\bar{\Lambda}^+, \bar{\Lambda}^+) = 2 k M \tag{II.11}$$

where both  $\bar{\mu}$  and  $\bar{\Lambda}^+$  are on the weight lattice of  $A_N$ . We will show in a second work that a similar **but not the same** relation governs the Weyl orbits of infinite dimensional Lie algebras beyond affine ones.

As is also emphasized in the book of Kac, (II.11) is of central importance for an explicit knowledge on the Weyl orbital structure of affine Lie algebras. The most economical way to solve (II.11) is to recall in general that  $A_N$  dominant weights  $\bar{\mu}^+$  which respect (II.11) are to be specified by

$$\bar{\mu}^+ = \bar{\Lambda}^+ + \sum_{i=1}^N r_i \alpha_i. \tag{II.12}$$

where  $r_i$ 's are some non-negative integers. It is seen then that for each and every order  $M$  of depth, there is only a finite number of sets  $\{r_1, r_2, \dots, r_N\}$ . Let us define  $\wp_M(\Lambda^+)$  as being the set of  $A_N$  dominant weights which correspond to these finite number of sets  $\{r_1, r_2, \dots, r_N\}$ . This definition is of central importance due to following lemma:

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(\*) the concept of sub-dominant weights come from the nomenclature of finite Lie algebras while they are called maximal weights in the notation of Kac

### Lemma

Any  $A_N$  dominant weight  $\bar{\mu}^+$  which solve (II.11) via (II.12) is always contained in  $\wp_M(\Lambda^+)$  if and only if  $\Lambda^+$  is equal to the one of affine fundamental dominant weights  $\Lambda_0, \Lambda_1, \dots, \Lambda_N$ . For any other

$$\Lambda^+ \equiv \Lambda_{\nu_1} + \Lambda_{\nu_2} ,$$

elements of  $\wp_M(\Lambda^+)$  can be chosen uniquely from the set

$$\wp_{M_1}(\Lambda_{\nu_1}) \oplus \wp_{M_2}(\Lambda_{\nu_2})$$

on condition that (II.11) is also fulfilled. This can be repeated similarly for any number of times.

We remark here that not all the weights of  $\wp_{M_1}(\Lambda_{\nu_1}) \oplus \wp_{M_2}(\Lambda_{\nu_2})$  fulfill (II.11) and hence it is seen that this lemma completely solves the problem of finding all the weights contributing in (I.1) or (I.4). One can now arrive at the conclusions that

- the depth M can only be a non-positive integer,
- at each and every order M of depth, there can only be a finite number of dominant weights  $\bar{\mu}^+$ , i.e.  $\dim \wp_M(\Lambda^+) < \infty$ .

The following observation would however be of great help here. Let us rewrite (II.11) for (II.12) together with

$$\bar{\Lambda}^+ \equiv \sum_{i=1}^N p_i \bar{\lambda}_i \quad (II.13)$$

where  $p_i$ 's are some non-negative integers. (II.11) will then turns out to be the following algebraic equation:

$$\sum_{i=1}^N r_i (r_i + p_i) - \sum_{i=1}^{N-1} r_i r_{i+1} = k M . \quad (II.14)$$

We finally conclude that:

Let  $\Lambda$  be an affine dominant or strictly dominant weight for which  $\sigma(\Lambda) = k \Lambda_0 + M \delta + \bar{\mu}$  where  $\sigma$  is an affine Weyl reflection. Let also  $\sigma^+(\Lambda) = k \Lambda_0 + M \delta + \bar{\mu}^+$  where  $\sigma^+$  is in general another affine Weyl reflection and  $\bar{\mu}^+ \in \wp_M(\Lambda)$ . Then the weights  $\bar{\mu}$  are in one-to-one correspondence with elements of finite Weyl orbits  $W(\bar{\mu}^+)$ . This in essence means that infinite summations over affine Weyl groups of  $A_N^{(1)}$  Lie algebras can be expressed in terms of finite Weyl groups of  $A_N$  Lie algebras. It is already known that, **in the fundamental weight basis**, actions of  $A_N$  Weyl groups can always be represented by ordinary permutations.

The second problem is to determine the signatures  $\epsilon(\mu)$  where  $\mu \in W(\Lambda^{++})$ . In the light of following specifications, this problem is to be solved in a quite non-trivial way. Let, for  $i = 1, 2, \dots, N$ ,  $n_i$ 's be the set of non-negative integers and  $s_i$ 's be the set of integers taking their values from the finite set  $\{0, 1, 2, \dots, k\}$  for any positive integer k. As is given in above lemma, let also that  $\bar{\mu}^+$  be an  $A_N$  dominant weight which solves

$$(\bar{\mu}^+, \bar{\mu}^+) - (\bar{\Lambda}^{++}, \bar{\Lambda}^{++}) = 2 k M$$

in such a way that

$$\sigma^{++}(\Lambda^{++}) = k \Lambda_0 + M \delta + \bar{\mu}^+$$

where  $\sigma^{++}$  is an affine Weyl reflection. If one now considers the decomposition

$$\bar{\mu}^+ \equiv \sum_{i=1}^N (s_i + (N+1) n_i) \mu_i , \quad (II.15)$$

the index defined on the right-hand side of the following expression will give the correct signatures:

$$\epsilon(\bar{\mu}^+) = \epsilon(s_1, s_2, \dots, s_N) \prod_{i=1}^N (-1)^{n_i} \quad (II.16)$$

where  $\epsilon(s_1, s_2, \dots, s_N)$  's are completely antisymmetric in their indices and also

$$\epsilon(s_1, s_2, \dots, s_N) \equiv 1 \quad , \quad s_1 \geq s_2 \geq \dots \geq s_N \quad . \quad (II.17)$$

For any other Weyl conjugates  $\bar{\mu}$  of  $\bar{\mu}^+$  there are additional contributions to (II.16) from the signatures of ordinary permutations. This, in fact, shows how Schur functions enter in the calculations of characters of  $A_N$  Lie algebras.

In result, the problems mentioned in the introduction are now being solved. In the next section, we will show that how can we use these results in applications of Weyl-Kac character formula. As we will show in a subsequent publication, it would be interesting to see how this also opens up a route to systematic applications of Weyl-Kac character formula for infinite dimensional Lie algebras beyond affine Lie algebras.

### III. CALCULATION OF STRING FUNCTIONS

In view of preceding section, we now explain how one makes use of the existence of permutation weights in an explicit calculation of (I.1) and (I.2). Most of our discussions, concerning  $A_N$  Lie algebras, has been given in detail in an unpublished work [12] though some of them will also be emphasized in the following.

For (I.1), we have two points to explain:

- the multiplicities  $c_{\mu, \Lambda^+}(M)$  are invariant under Weyl group actions, i.e. they have the same value for any  $\mu$  having the same depth M ,
- in the specialization  $e^{\mu_I} = u_I$ , characters can be defined by

$$ChW(\bar{\mu}^+) = K_{q_1, q_2, \dots, q_s}(u_1, u_2, \dots, u_N) \quad (III.1)$$

for the whole Weyl orbit  $W(\bar{\mu}^+)$  of an  $A_N$  dominant weight

$$\bar{\mu}^+ \equiv \sum_{i=1}^s q_i \mu_i \quad , \quad q_1 \geq q_2 \geq \dots \geq q_s > 0 \quad (III.2)$$

where  $s \leq N$  and

$$K_{q_1, q_2, \dots, q_N}(u_1, u_2, \dots, u_N) \equiv \sum_{j_1, j_2, \dots, j_s=1}^{N+1} (u_{j_1})^{q_1} \dots (u_{j_s})^{q_s} . \quad (III.3)$$

These polinomials can be reduced to class functions

$$K_q(u_1, u_2, \dots, u_N) \equiv \sum_{j=1}^{N+1} (u_j)^q \quad (III.4)$$

via some reduction rules. Note here also that the parameters  $u_I$  are constrained by

$$\prod_{I=1}^{N+1} u_I = 1 \quad (III.5)$$

as a result of (II.3).

In ref.12, a detailed calculation of (I.2) is given for  $A_N$  Lie algebras. In view of decomposition (III.2) of  $A_N$  dominant weights, we have, as a result of some reduction procedure,

$$A(\rho + \bar{\mu}^+) = A(\rho) S_{q_1, q_2, \dots, q_s}(x_1, x_2, \dots, x_N) \quad (III.6)$$

where

$$A(\rho) = \prod_{j>i=1}^{N+1} (u_i - u_j)$$

is the Vandermonde determinant and  $S_{q_1, q_2, \dots, q_s}(x_1, x_2, \dots, x_N)$ 's are Schur polynomials. The indeterminates  $x_s$  are defined here by the equivalence

$$K_q(u_1, u_2 \dots u_N) \equiv q x_q \quad , \quad q = 1, 2 \dots N \quad . \quad (III.7)$$

(III.7) provides an inter-relation between the sets  $\{u_1, u_2 \dots u_N\}$  and  $\{x_1, x_2 \dots x_N\}$  of indeterminates and has crucial importance to equate (I.1) and (I.2) in practical calculations. The reduction procedure to express Schur polynomials in terms of **classical Schur polynomials**  $S_q(x_1, x_2 \dots x_N)$  is known. We know that, for a fixed value of  $N$ , there are only  $N$  classical Schur polynomials  $S_q(x_1, x_2 \dots x_N)$  with  $q = 1, 2 \dots N$  though one could also need in general those of  $q > N$ . This arises the problem of extending classical Schur polynomials  $S_q(x_1, x_2 \dots x_N)$  for  $q > N$  and these are the so-called **degenerated Schur polynomials**. To this end, it is shown in ref.12 that the following reduction rules are sufficient:

$$S_q = (-1)^{N+1} S_{q-N} - \sum_{r=1}^{N+1} S_r^* S_{q-r} \quad , \quad q \geq N+1 \quad (III.8)$$

where  $S_q^*$  is obtained from  $S_q$  under replacements  $x_i \rightarrow -x_i$ .

We will now show that the calculations for affine Lie algebras are to be reduced to those of the finite ones if one uses permutation weights. To proceed further, it will be more instructive to follow an explicit example. A very typical one is, for instance, the level-2 representation  $R(\Lambda_0 + \Lambda_1)$  of  $A_5^{(1)}$ . Its sub-dominants are given by

$$\begin{aligned} M_0(\Lambda_0 + \Lambda_1, \Lambda_0 + \Lambda_1) &= 0, \\ M_0(\Lambda_2 + \Lambda_5, \Lambda_0 + \Lambda_1) &= 1, \\ M_0(\Lambda_3 + \Lambda_4, \Lambda_0 + \Lambda_1) &= 2. \end{aligned} \quad (III.9)$$

We thus have, from (I.5), three string functions

$$\begin{aligned} C_{\Lambda_0 + \Lambda_1}(\Lambda_0 + \Lambda_1) &\equiv \sum_{M=0}^{\infty} q^M c_{\Lambda_0 + \Lambda_1, \Lambda_0 + \Lambda_1}(M) \quad , \\ C_{\Lambda_0 + \Lambda_1}(\Lambda_2 + \Lambda_5) &\equiv \sum_{M=1}^{\infty} q^M c_{\Lambda_2 + \Lambda_5, \Lambda_0 + \Lambda_1}(M) \quad , \\ C_{\Lambda_0 + \Lambda_1}(\Lambda_3 + \Lambda_4) &\equiv \sum_{M=2}^{\infty} q^M c_{\Lambda_3 + \Lambda_4, \Lambda_0 + \Lambda_1}(M) \end{aligned} \quad (III.10)$$

where  $q \equiv e^{-\delta}$  is an indeterminate. Instead of multiplicities, if one simply puts these string functions in (I.1) one obtains

$$\begin{aligned} Ch(\Lambda_0 + \Lambda_1) &= C_{\Lambda_0 + \Lambda_1}(\Lambda_0 + \Lambda_1) \sum_{M=0}^{\infty} q^M \sum_{\bar{\mu}^+ \in \wp(\Lambda_0 + \Lambda_1, M)} K_{\bar{\mu}^+}(u_1, u_2 \dots u_5) + \\ &C_{\Lambda_0 + \Lambda_1}(\Lambda_2 + \Lambda_5) \sum_{M=1}^{\infty} q^M \sum_{\bar{\mu}^+ \in \wp(\Lambda_2 + \Lambda_5, M)} K_{\bar{\mu}^+}(u_1, u_2 \dots u_5) + \\ &C_{\Lambda_0 + \Lambda_1}(\Lambda_3 + \Lambda_4) \sum_{M=2}^{\infty} q^M \sum_{\bar{\mu}^+ \in \wp(\Lambda_3 + \Lambda_4, M)} K_{\bar{\mu}^+}(u_1, u_2 \dots u_5) . \end{aligned} \quad (III.11)$$

For any  $K=0, 1, 2, \dots$ , we formally define here

$$\wp(\Lambda, K) \equiv \bigcup_{M=0}^K \wp_M(\Lambda). \quad (III.12)$$

where  $\bigcup$  means simple collection. We also find useful to adopt the notation

$$K_{q_1, q_2 \dots q_5}(u_1, u_2 \dots u_5) \equiv K_{\bar{\mu}^+}(u_1, u_2 \dots u_5) \quad (III.13)$$

in the light of (III.2). As is emphasized above, the reduction rules given in ref.12 allow us to reduce  $K_{\bar{\mu}^+}(u_1, u_2 \dots u_5)$ 's in terms of class functions  $K_q(u_1, u_2 \dots u_5)$ 's. In result, (III.11) can be expressed in the following serie expansion:

$$Ch(\Lambda_0 + \Lambda_1) \simeq \sum_{J=0}^K q^J LEFT_J(u_1, u_2 \dots u_5) . \quad (III.14)$$

On the other hand, the simple factorization in (III.6) does not occur if one wants to apply (I.2) for affine Lie algebras. Note that (III.6) is valid only for finite Lie algebras and for infinite dimensional Lie algebras one could only expect to obtain the string functions with only positive integer coefficients, i.e. positive integer multiplicities. As in (III.13), let us also assume that

$$S_{q_1, q_2 \dots q_5}(x_1, x_2 \dots x_5) \equiv S_{\bar{\mu}^+}(x_1, x_2 \dots x_5) . \quad (III.15)$$

In view of (A.9) and (A.10), following results will then be obtained from (I.4):

$$\begin{aligned} A(\tilde{\rho}, K) &= A(\rho) \sum_{\bar{\mu}^+ \in \wp(\tilde{\rho}, K)} S_{\bar{\mu}^+ - \rho}(x_1, x_2 \dots x_5) , \\ A(\tilde{\rho} + \Lambda_0 + \Lambda_1, K) &= A(\rho) \sum_{\bar{\mu}^+ \in \wp(\tilde{\rho} + \Lambda_0 + \Lambda_1, K)} S_{\bar{\mu}^+ - \rho}(x_1, x_2 \dots x_5) \end{aligned} \quad (III.16)$$

where  $A(\rho)$  is still the Vandermonde determinant defined above. It is clear that this allows us to make the serie expansion

$$\frac{A(\tilde{\rho} + \Lambda_0 + \Lambda_1, K)}{A(\tilde{\rho}, K)} \simeq \sum_{J=0}^K q^J RIGHT_J(x_1, x_2 \dots x_5) \quad (III.17)$$

which determine (I.2) up to any order  $K$  ( $=0, 1 \dots$ ).

The experienced reader knows here that actual calculations bring out severe difficulties in practice. It will therefore be quite suitable to consider the specialization

$$u_1 = \kappa, \quad u_2 = \kappa^{-1}, \quad u_3 = u_4 = u_5 = 1 \quad (III.18)$$

for which one has, as a result of (III.7),

$$x_i = \frac{4}{i} + \frac{\kappa}{i} + \frac{\kappa^{-1}}{i} \quad (III.19)$$

for all values of  $i$ , i.e.  $i \leq 5$  or  $i > 5$ . With this remark in mind, the equality for the right-hand sides of (III.14) and (III.17) is always valid for any fixed value of  $K$  and this is the Weyl-Kac formula which provides the string functions in (III.10) up to order 9:

$$\begin{aligned} C_{\Lambda_0 + \Lambda_1}(\Lambda_0 + \Lambda_1) &= 1 + 10 q + 70 q^2 + 380 q^3 + 1740 q^4 + \\ &\quad 7012 q^5 + 25585 q^6 + 86130 q^7 + 271225 q^8 + 807100 q^9 + \dots \end{aligned}$$

$$\begin{aligned} C_{\Lambda_0 + \Lambda_1}(\Lambda_2 + \Lambda_5) &= q (2 + 22 q + 148 q^2 + 770 q^3 + 3382 q^4 + \\ &\quad 13134 q^5 + 46382 q^6 + 151734 q^7 + 465894 q^8 + \dots) \end{aligned}$$

$$C_{\Lambda_0 + \Lambda_1}(\Lambda_3 + \Lambda_4) = q^2 (5 + 50 q + 315 q^2 + 1550 q^3 + 6506 q^4 + 24320 q^5 + 83140 q^6 + 264460 q^7 + \dots) .$$

All of these multiplicities will be obtained from equations which appear to be coefficients of powers of the free parameter  $\kappa$  in the Weyl-Kac formula and it will be seen that the calculations need very restricted computer times, say, in Mathematica [13].

## CONCLUSIONS

We have two conclusions one of which is for the present and the other is for a future work. As is seen in above example, the equality of (III.14) and (III.17) is valid for any order  $K$  of depth on condition that the corresponding sets  $\wp(\Lambda, K)$  of permutation weights are known for necessary affine dominant weights  $\Lambda$ 's. And the whole machinery which developed in this work for affine Lie algebras can be applied, with only an appropriate modification, beyond affine Lie algebras [14].



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## APPENDIX

In this appendix, the notation

$$M \delta + \sum_{i=1}^N p_i \mu_i \equiv (p_1, p_2 \dots p_N)_{-M} \quad (A.1)$$

will be useful with the emphasis that all weights in any  $\wp(\Lambda, K)$  will be of the same level with  $\Lambda$ . The complete data for permutation weights which occur to find all the multiplicities encountered in the calculation of  $Ch(\Lambda_0 + \Lambda_1)$  up to ninth order will be given.

All affine fundamental dominant weights are of level-1 and their permutation weights are as in the following:

$$\begin{aligned} \wp(\Lambda_0, 9) = \{ & (0, 0, 0, 0, 0)_0 , \\ & (2, 1, 1, 1, 1)_1 , \\ & (2, 2, 1, 1, 0)_2 , \\ & (3, 3, 2, 2, 2)_3 , (2, 2, 2, 0, 0)_3 , (3, 1, 1, 1, 0)_3 , \\ & (3, 3, 3, 2, 1)_4 , (4, 2, 2, 2, 2)_4 , (3, 2, 1, 0, 0)_4 , \\ & (4, 3, 2, 2, 1)_5 , \\ & (4, 4, 4, 3, 3)_6 , (3, 3, 3, 3, 0)_6 , \\ & (4, 3, 3, 1, 1)_6 , (3, 3, 0, 0, 0)_6 , (4, 1, 1, 0, 0)_6 , \\ & (4, 4, 4, 4, 2)_7 , (5, 4, 3, 3, 3)_7 , (4, 3, 3, 2, 0)_7 , \\ & (4, 4, 2, 1, 1)_7 , (5, 2, 2, 2, 1)_7 , (4, 2, 0, 0, 0)_7 , \\ & (5, 4, 4, 3, 2)_8 , (4, 4, 2, 2, 0)_8 , (5, 3, 2, 1, 1)_8 , \\ & (5, 5, 3, 3, 2)_9 , (4, 4, 3, 1, 0)_9 , (6, 3, 3, 3, 3)_9 , (5, 3, 2, 2, 0)_9 \} \end{aligned} \quad (A.2)$$

$$\begin{aligned}
\wp(\Lambda_1, 9) = \{ & (1, 0, 0, 0, 0)_0 , \\
& (2, 2, 1, 1, 1)_1 , \\
& (2, 2, 2, 1, 0)_2 , (3, 1, 1, 1, 1)_2 , \\
& (3, 3, 3, 2, 2)_3 , (3, 2, 1, 1, 0)_3 , \\
& (3, 3, 3, 3, 1)_4 , (4, 3, 2, 2, 2)_4 , (3, 2, 2, 0, 0)_4 , \\
& (4, 3, 3, 2, 1)_5 , (3, 3, 1, 0, 0)_5 , (4, 1, 1, 1, 0)_6 , (4, 4, 4, 4, 3)_6 , \\
& (4, 4, 2, 2, 1)_6 , (5, 2, 2, 2, 2)_6 , (4, 2, 1, 0, 0)_6 , \\
& (5, 4, 4, 3, 3)_7 , (4, 3, 3, 3, 0)_7 , (4, 4, 3, 1, 1)_7 , (5, 3, 2, 2, 1)_7 , \\
& (5, 4, 4, 4, 2)_8 , (5, 5, 3, 3, 3)_8 , \\
& (4, 4, 3, 2, 0)_8 , (5, 3, 3, 1, 1)_8 , (4, 3, 0, 0, 0)_8 , \\
& (5, 5, 4, 3, 2)_9 , (6, 4, 3, 3, 3)_9 , \\
& (5, 3, 3, 2, 0)_9 , (5, 4, 2, 1, 1)_9 , (5, 1, 1, 0, 0)_9 \}
\end{aligned} \tag{A.3}$$

$$\begin{aligned}
\wp(\Lambda_2, 9) = \{ & (1, 1, 0, 0, 0)_0 , \\
& (2, 2, 2, 1, 1)_1 , (2, 0, 0, 0, 0)_1 , \\
& (2, 2, 2, 2, 0)_2 , (3, 2, 1, 1, 1)_2 , \\
& (3, 3, 3, 3, 2)_3 , (3, 2, 2, 1, 0)_3 , \\
& (4, 3, 3, 2, 2)_4 , (3, 3, 1, 1, 0)_4 , (4, 1, 1, 1, 1)_4 , \\
& (4, 3, 3, 3, 1)_5 , (4, 4, 2, 2, 2)_5 , (3, 3, 2, 0, 0)_5 , (4, 2, 1, 1, 0)_5 , \\
& (4, 4, 4, 4, 4)_6 , (4, 4, 3, 2, 1)_6 , (5, 3, 2, 2, 2)_6 , (4, 2, 2, 0, 0)_6 , \\
& (5, 4, 4, 4, 3)_7 , (5, 3, 3, 2, 1)_7 , (4, 3, 1, 0, 0)_7 , \\
& (5, 5, 4, 3, 3)_8 , (4, 4, 3, 3, 0)_8 , \\
& (4, 4, 4, 1, 1)_8 , (5, 4, 2, 2, 1)_8 , (5, 1, 1, 1, 0)_8 , \\
& (5, 5, 4, 4, 2)_9 , (4, 4, 4, 2, 0)_9 , (6, 4, 4, 3, 3)_9 , \\
& (5, 3, 3, 3, 0)_9 , (5, 4, 3, 1, 1)_9 , (6, 2, 2, 2, 2)_9 , (5, 2, 1, 0, 0)_9 \}
\end{aligned} \tag{A.4}$$

$$\begin{aligned}
\wp(\Lambda_3, 9) = \{ & (1, 1, 1, 0, 0)_0 , \\
& (2, 2, 2, 2, 1)_1 , (2, 1, 0, 0, 0)_1 , \\
& (3, 2, 2, 1, 1)_2 , \\
& (3, 3, 3, 3, 3)_3 , (3, 2, 2, 2, 0)_3 , (3, 3, 1, 1, 1)_3 , (3, 0, 0, 0, 0)_3 , \\
& (4, 3, 3, 3, 2)_4 , (3, 3, 2, 1, 0)_4 , (4, 2, 1, 1, 1)_4 , \\
& (4, 4, 3, 2, 2)_5 , (4, 2, 2, 1, 0)_5 , \\
& (4, 4, 3, 3, 1)_6 , (3, 3, 3, 0, 0)_6 , (5, 3, 3, 2, 2)_6 , (4, 3, 1, 1, 0)_6 , \\
& (5, 4, 4, 4, 4)_7 , (4, 4, 4, 2, 1)_7 , (5, 3, 3, 3, 1)_7 , \\
& (5, 4, 2, 2, 2)_7 , (4, 3, 2, 0, 0)_7 , (5, 1, 1, 1, 1)_7 , \\
& (5, 5, 4, 4, 3)_8 , (5, 4, 3, 2, 1)_8 , (5, 2, 1, 1, 0)_8 , \\
& (5, 5, 5, 3, 3)_9 , (4, 4, 4, 3, 0)_9 , (6, 4, 4, 4, 3)_9 , \\
& (4, 4, 1, 0, 0)_9 , (6, 3, 2, 2, 2)_9 , (5, 2, 2, 0, 0)_9 \}
\end{aligned} \tag{A.5}$$

It is seen that  $\wp(\Lambda_0, 9)$  and  $\wp(\Lambda_3, 9)$  are real under  $A_N$  diagram automorphism while  $\wp(\Lambda_4, 9)$  and  $\wp(\Lambda_5, 9)$  will be conjugates, respectively, to  $\wp(\Lambda_2, 9)$  and  $\wp(\Lambda_1, 9)$ .

We know from above, we have three sub-dominant weight of level-2 for  $Ch(\Lambda_0 + \Lambda_1)$ . Among elements

of the set  $\wp_{M_1}(\Lambda_0) \oplus \wp_{M_2}(\Lambda_1)$ , the ones which fulfill (II.11) form

$$\begin{aligned}
\wp(\Lambda_0 + \Lambda_1, 9) = \{ & (1, 0, 0, 0, 0)_0 , \\
& (3, 1, 1, 1, 1)_1 , \\
& (4, 3, 2, 2, 2)_2 , \\
& (4, 4, 2, 2, 1)_3 , \quad (5, 2, 2, 2, 2)_3 , \\
& (4, 4, 3, 2, 0)_4 , \quad (5, 5, 3, 3, 3)_4 , \\
& (4, 4, 4, 1, 0)_5 , \quad (5, 4, 2, 2, 0)_5 , \quad (6, 2, 2, 2, 1)_5 , \\
& (5, 5, 5, 3, 1)_6 , \quad (6, 3, 2, 2, 0)_6 , \quad (6, 6, 5, 4, 4)_6 , \quad (7, 3, 3, 3, 3)_6 , \\
& (5, 4, 4, 0, 0)_7 , \quad (6, 4, 2, 1, 0)_7 , \quad (6, 6, 6, 4, 3)_7 , \quad (7, 6, 4, 4, 4)_7 , \\
& (6, 4, 3, 0, 0)_8 , \quad (6, 6, 6, 5, 2)_8 , \\
& (7, 2, 2, 2, 0)_8 , \quad (7, 5, 3, 3, 1)_8 , \quad (8, 5, 4, 4, 4)_8 , \\
& (6, 5, 2, 0, 0)_9 , \quad (7, 6, 6, 4, 2)_9 , \quad (7, 7, 7, 5, 5)_9 , \quad (8, 6, 4, 4, 3)_9 \}
\end{aligned} \tag{A.6}$$

and similarly one has

$$\begin{aligned}
\wp(\Lambda_2 + \Lambda_5, 9) = \{ & (2, 2, 1, 1, 1)_0 , \\
& (3, 2, 1, 1, 0)_1 , \quad (3, 3, 3, 2, 2)_1 , \\
& (3, 3, 1, 0, 0)_2 , \quad (4, 1, 1, 1, 0)_2 , \quad (4, 3, 3, 2, 1)_2 , \\
& (4, 3, 3, 3, 0)_3 , \quad (4, 4, 3, 1, 1)_3 , \quad (5, 3, 2, 2, 1)_3 , \quad (5, 4, 4, 3, 3)_3 , \\
& (5, 1, 1, 0, 0)_4 , \quad (5, 3, 3, 2, 0)_4 , \\
& (5, 4, 2, 1, 1)_4 , \quad (5, 5, 4, 3, 2)_4 , \quad (6, 4, 3, 3, 3)_4 , \\
& (5, 4, 3, 1, 0)_5 , \quad (5, 5, 4, 4, 1)_5 , \quad (5, 5, 5, 2, 2)_5 , \\
& (6, 3, 2, 1, 1)_5 , \quad (6, 5, 3, 3, 2)_5 , \quad (6, 5, 5, 5, 4)_5 , \\
& (5, 5, 2, 1, 0)_6 , \quad (6, 3, 3, 1, 0)_6 , \quad (6, 4, 1, 1, 1)_6 , \\
& (6, 5, 4, 3, 1)_6 , \quad (6, 6, 5, 5, 3)_6 , \quad (7, 4, 3, 3, 2)_6 , \quad (7, 5, 5, 4, 4)_6 , \\
& (5, 5, 3, 0, 0)_7 , \quad (5, 5, 5, 4, 0)_7 , \quad (6, 5, 5, 2, 1)_7 , \quad (6, 6, 3, 3, 1)_7 , \\
& (7, 2, 2, 1, 1)_7 , \quad (7, 4, 4, 3, 1)_7 , \quad (7, 5, 3, 2, 2)_7 , \quad (7, 6, 5, 4, 3)_7 , \\
& (6, 5, 1, 1, 0)_8 , \quad (6, 5, 5, 3, 0)_8 , \quad (7, 3, 2, 1, 0)_8 , \\
& (7, 5, 4, 2, 1)_8 , \quad (7, 6, 5, 5, 2)_8 , \quad (7, 6, 6, 3, 3)_8 , \\
& (7, 7, 4, 4, 3)_8 , \quad (7, 7, 6, 6, 5)_8 , \quad (8, 3, 3, 3, 2)_8 , \quad (8, 5, 5, 4, 3)_8 , \\
& (6, 6, 5, 1, 1)_9 , \quad (7, 3, 3, 0, 0)_9 , \quad (7, 4, 1, 1, 0)_9 , \\
& (7, 5, 4, 3, 0)_9 , \quad (7, 6, 3, 2, 1)_9 , \quad (7, 7, 5, 4, 2)_9 , \quad (7, 7, 7, 6, 4)_9 , \\
& (8, 4, 3, 3, 1)_9 , \quad (8, 5, 5, 5, 2)_9 , \quad (8, 6, 5, 3, 3)_9 , \quad (8, 7, 6, 5, 5)_9 \}
\end{aligned} \tag{A.7}$$

$$\begin{aligned}
\wp(\Lambda_3 + \Lambda_4, 9) = \{ & (2, 2, 2, 1, 0)_0 , \\
& (3, 2, 2, 0, 0)_1 , (3, 3, 3, 3, 1)_1 , \\
& (4, 2, 1, 0, 0)_2 , (4, 4, 4, 4, 3)_2 , \\
& (4, 3, 0, 0, 0)_3 , (5, 3, 3, 1, 1)_3 , (5, 4, 4, 4, 2)_3 , \\
& (5, 2, 0, 0, 0)_4 , (5, 5, 5, 5, 5)_4 , (6, 4, 4, 3, 2)_4 , \\
& (5, 5, 1, 1, 1)_5 , (6, 4, 4, 4, 1)_5 , (6, 5, 4, 2, 2)_5 , \\
& (6, 1, 0, 0, 0)_6 , (6, 6, 3, 2, 2)_6 , (7, 4, 4, 2, 2)_6 , (7, 5, 5, 5, 3)_6 , \\
& (6, 5, 4, 4, 0)_7 , (6, 6, 4, 2, 1)_7 , (7, 3, 1, 1, 1)_7 , (7, 6, 6, 6, 6)_7 , \\
& (6, 6, 4, 3, 0)_8 , (7, 4, 4, 4, 0)_8 , (7, 6, 2, 2, 2)_8 , \\
& (7, 7, 5, 3, 3)_8 , (8, 4, 3, 2, 2)_8 , (8, 6, 6, 6, 5)_8 , \\
& (6, 6, 5, 2, 0)_9 , (7, 0, 0, 0, 0)_9 , \\
& (8, 4, 4, 2, 1)_9 , (8, 5, 2, 2, 2)_9 , (8, 7, 6, 6, 4)_9 \} .
\end{aligned} \tag{A.8}$$

The equivalent of  $Ch(\Lambda_0 + \Lambda_1)$  is provided by the Weyl-Kac formula. For this, we first define affine Weyl vector  $\tilde{\rho}$  as in the following:

$$\tilde{\rho} \equiv \sum_{\nu=0}^N \Lambda_{\nu} .$$

By applying successively above lemma 6 times in any order, following decomposition for  $\wp(\tilde{\rho}, 9)$  will then be obtained:

$$\begin{aligned}
\wp(\tilde{\rho}, 9) = \{ & (5, 4, 3, 2, 1)_0 , \\
& (7, 5, 4, 3, 2)_1 , \\
& (8, 5, 4, 3, 1)_2 , (8, 7, 5, 4, 3)_2 , \\
& (9, 5, 4, 2, 1)_3 , (9, 7, 5, 4, 2)_3 , (9, 8, 7, 5, 4)_3 , \\
& (10, 5, 3, 2, 1)_4 , (10, 7, 5, 3, 2)_4 , \\
& (10, 8, 7, 5, 3)_4 , (9, 8, 5, 4, 1)_4 , (10, 9, 8, 7, 5)_4 , \\
& (11, 4, 3, 2, 1)_5 , (11, 7, 4, 3, 2)_5 , (11, 8, 7, 4, 3)_5 , \\
& (10, 8, 5, 3, 1)_5 , (11, 9, 8, 7, 4)_5 , (10, 9, 7, 5, 2)_5 , (11, 10, 9, 8, 7)_5 , \\
& (10, 9, 5, 2, 1)_6 , (11, 8, 4, 3, 1)_6 , \\
& (11, 9, 7, 4, 2)_6 , (11, 10, 8, 7, 3)_6 , (10, 9, 8, 5, 1)_6 , \\
& (13, 11, 10, 9, 8)_7 , (11, 9, 4, 2, 1)_7 , (11, 10, 7, 3, 2)_7 , \\
& (13, 10, 9, 8, 5)_7 , (13, 9, 8, 5, 4)_7 , (13, 8, 5, 4, 3)_7 , \\
& (13, 5, 4, 3, 2)_7 , (11, 9, 8, 4, 1)_7 , (11, 10, 9, 7, 2)_7 , \\
& (14, 11, 10, 9, 7)_8 , (11, 10, 3, 2, 1)_8 , (11, 10, 8, 3, 1)_8 , \\
& (14, 10, 9, 7, 5)_8 , (13, 11, 9, 8, 4)_8 , (14, 9, 7, 5, 4)_8 , \\
& (13, 10, 8, 5, 3)_8 , (13, 9, 5, 4, 2)_8 , (14, 7, 5, 4, 3)_8 , (11, 10, 9, 8, 1)_8 , \\
& (15, 11, 10, 8, 7)_9 , (13, 11, 8, 4, 3)_9 , (13, 10, 5, 3, 2)_9 , \\
& (14, 13, 11, 10, 9)_9 , (11, 10, 9, 2, 1)_9 , (15, 10, 8, 7, 5)_9 , \\
& (14, 11, 9, 7, 4)_9 , (13, 11, 10, 8, 3)_9 , (14, 10, 7, 5, 3)_9 , \\
& (15, 8, 7, 5, 4)_9 , (13, 10, 9, 5, 2)_9 , (14, 5, 4, 3, 1)_9 \} .
\end{aligned} \tag{A.9}$$

One also has similarly

$$\begin{aligned}
\wp(\tilde{\rho} + \Lambda_0 + \Lambda_1, 9) = \{ & (6, 4, 3, 2, 1)_0 , \\
& (0, 0, 0, 0, 0)_1 , \\
& (10, 6, 5, 4, 3)_2 , \\
& (11, 6, 5, 4, 2)_3 , \\
& (12, 10, 7, 6, 5)_4 , \ (12, 6, 5, 3, 2)_4 , \\
& (13, 10, 7, 6, 4)_5 , \ (13, 11, 9, 7, 6)_5 , \ (13, 6, 4, 3, 2)_5 , \\
& (14, 10, 7, 5, 4)_6 , \ (14, 11, 9, 7, 5)_6 , \ (14, 12, 10, 9, 7)_6 , \ (14, 5, 4, 3, 2)_6 , \\
& (13, 12, 7, 6, 2)_7 , \ (15, 10, 6, 5, 4)_7 , \\
& (15, 11, 9, 6, 5)_7 , \ (15, 12, 10, 9, 6)_7 , \ (15, 13, 11, 10, 9)_7 , \\
& (14, 12, 7, 5, 2)_8 , \ (14, 13, 9, 7, 3)_8 , \\
& (14, 13, 7, 4, 2)_9 , \ (15, 12, 6, 5, 2)_9 , \\
& (15, 13, 9, 6, 3)_9 , \ (15, 14, 10, 9, 4)_9 , \ (17, 14, 12, 11, 10)_9 , \\
& (17, 13, 11, 10, 7)_9 , \ (17, 12, 10, 7, 6)_9 , \ (17, 11, 7, 6, 5)_9 \} .
\end{aligned} \tag{A.10}$$